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## HYPOELLIPTIC OPERATORS OF PRINCIPAL TYPE WITH INFINITE DEGENERACY

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday in 1992

By

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### Introduction.

Let  $P$  be a classical pseudodifferential operator of order  $m$ . We assume  $P$  is of principal type, that is, the Hamilton vector field  $H_p$  of the principal symbol  $p$  of  $P$  is not parallel to the radial direction where the principal symbol  $p$  vanishes. In this paper we study the microhypoellipticity for  $P$ , under the following  $(\bar{\Psi})$  condition given by Nirenberg-Treves [15];

$$(\bar{\Psi}) \quad \begin{cases} \text{the imaginary part } p_2 \text{ of the principal symbol } p \\ \text{does not change sign from } + \text{ to } - \text{ along any oriented} \\ \text{(null-) bicharacteristic of the real part } p_1 \text{ of } p. \end{cases}$$

Let us recall that  $(\bar{\Psi})$  is necessary for adjoint operator  $P^*$  of  $P$  to be locally solvable (see Hörmander [6; Theorem 26.4.7], cf. Moyer [14]). Since it follows from the hypoellipticity of  $P$  that  $P^*$  is locally solvable, it is reasonable to assume the condition  $(\bar{\Psi})$ .

By supplying the missing arguments of Egorov [2], Hörmander [5] (see also [6; Chapter 27]) showed that a pseudodifferential operator  $P$  of principal type is subelliptic (and hence hypoelliptic) if and only if the principal symbol  $p$  of  $P$  satisfies  $(\bar{\Psi})$  and a finite type assumption ((27.1.8) in [6]). Without the finite type assumption, the problem of hypoellipticity seems to be difficult. For example, consider a first-order pseudodifferential operator of Egorov type as follows:

$$P_0 = D_t + i(t^s D_{x_1} + t^k x_1^m |D|) \quad \text{in } \mathbf{R}_t \times \mathbf{R}_x^n, \quad |D|^2 = D_t^2 + |D_x|^2,$$

where  $s, k, m$  are nonnegative integers. For  $P_0$ , condition  $(\bar{\Psi})$  and the finite type assumption are expressed as

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$$s, m \text{ even, } k \text{ odd, } s < k.$$

Then  $P_0$  is subelliptic with loss of  $r/(r+1)$  derivatives ( $r=k+m(s+1)$ ) and hence hypoelliptic. If  $t^s, t^k, x_1^m$  of  $P_0$  are replaced by functions infinitely vanishing then the hypoellipticity of  $P_0$  is unknown. The aim of the present paper is to solve this particular problem, but we shall reply only for special cases, unfortunately, because we do not know even whether  $L^2$  *a priori-estimate* holds for this modified  $P_0$ , in general. Actually, a remarkable counter-example given by Lerner [10] shows that we can not always expect  $L^2$  *a priori-estimate* for operators satisfying  $(\bar{\Psi})$ .

To end the introduction, we state a few historical remarks: As a perfection of the preceding results of Nirenberg-Treves [15] in the analytic case or the finite type case, Beals-Fefferman [1] proved  $L^2$  *a priori-estimate* (and hence local solvability) for pseudodifferential operators of principal type, under condition  $(P)$  (i.e. the imaginary part  $p_2$  of the principal symbol of  $P$  does not change sign along the bicharacteristic of the real part  $p_1$ , which is equivalent to  $(\bar{\Psi})$  for differential operators.) Furthermore, Hörmander [6; Chapter 26] extended the local existence result of [1] to the semi-global one and fully studied the regularities of solutions for operators, of principal type, satisfying condition  $(P)$ . Under condition  $(\bar{\Psi})$ ,  $L^2$  *a priori-estimate* for operators in 2-dimension space was proved by Lerner [8], whose method also plays an important role in the present paper.

## 1. Main results

Let  $P$  be a classical pseudodifferential operator on  $\mathbf{R}^{n+1}$ , of order  $m$ , of principal type, which satisfies the condition  $(\bar{\Psi})$ . We are interested in the micro-hypoellipticity of  $P$ ; that is, for  $\rho_0 \in T^*(\mathbf{R}^{n+1}) \setminus 0$ , we shall see whether

$$(1.1) \quad \rho_0 \notin \text{WF}(Pu) \text{ implies } \rho_0 \notin \text{WF}(u) \quad \text{for } \forall u \in \mathcal{D}'(\mathbf{R}^{n+1}).$$

We assume  $\rho_0 \in \text{Char } P$  because (1.1) is trivial, otherwise, where  $\text{Char } P$  denotes the set of characteristic points. Let  $p = p_1 + ip_2$  ( $p_1, p_2$  real-valued) be the principal symbol of  $P$  and let  $\Gamma$  be a subset of  $\text{Char } P$  where the Poisson bracket  $\{p_1, p_2\}$  vanishes. It is known by Hörmander's classical theorem [4] (and also Egorov-Hörmander Theorem [6; Theorem 27.1.11]) that (1.1) is true if  $\rho_0 \notin \Gamma$ , because we have a subelliptic estimate with loss of  $1/2$  derivatives. In what follows we consider the case where  $\rho_0 \in \Gamma$ . We assume that in a conic neighborhood of  $\rho_0$

$$(1.2) \quad \begin{cases} \Gamma \text{ is contained in a } C^\infty\text{-hypersurface in } T^*(\mathbf{R}^{n+1}) \setminus 0 \\ \text{to which the Hamilton vector field } H_1 \text{ of } p_1 \text{ is transversal.} \end{cases}$$

After the multiplication by an elliptic factor, we may assume  $P$  is of first order. Furthermore, by homogeneous canonical transformation and Malgrange preparation theorem we may assume that  $\rho_0 = (0, (0, \xi_0)) \in T^*(\mathbf{R}_t \times \mathbf{R}_x^n \setminus 0)$ , ( $|\xi_0| = 1$ ), and the principal symbol  $p$  of  $P$  is expressed as, in a small conic neighborhood  $V$  of  $\rho_0$ ,

$$(1.3) \quad p = p(t, x, \tau, \xi) = \tau + iq(t, x, \xi),$$

where  $q(t, x, \xi) \in C^\infty(\mathbf{R}_t \times \mathbf{R}_x^n \times \mathbf{R}_\xi^n)$  is real valued, positively homogeneous of degree one for  $|\xi| \geq 1/2$ ; in particular  $q$  satisfies:

$$(1.4) \quad q(t, x, \xi) = \lambda q(t, x, \xi/\lambda), \quad \text{if } |\xi| \geq 1/2 \text{ and } 0 < \lambda \leq 1,$$

and

$$(1.5) \quad |(D_t^k D_x^\alpha D_\xi^\beta q)(t, x, \xi)| \leq C_{\alpha, \beta, k} (1 + |\xi|)^{1 - |\beta|}.$$

We may also assume that lower order terms  $p_0, p_{-1}, \dots$  in the symbol of  $P$  are independent of  $\tau$  in a conic neighborhood  $V$  of  $\rho_0$  (see the paragraph after [6; Theorem 26.4.7']). Hence we can write

$$(1.3)' \quad P = D_t + iQ(t, x, D_x) \quad \text{in } V,$$

where the principal symbol of  $Q$  is  $q(t, x, \xi)$ . In that frame work, condition  $(\bar{\Psi})$  is expressed as

$$(1.6) \quad q(t, x, \xi) > 0 \text{ and } s > t \text{ imply } q(s, x, \xi) \geq 0.$$

Moreover, the set  $\Gamma$  is defined by

$$\{(t, x, 0, \xi) \in T^*(\mathbf{R}^{n+1}) \setminus 0; \partial_t q(t, x, \xi) = q(t, x, \xi) = 0\}$$

and it follows from assumption (1.2) that

$$(1.7) \quad \begin{cases} \text{for any } \mu > 0 \text{ there exists a } \delta_\mu > 0 \text{ such that} \\ \left\{ (t, x, 0, \xi); \mu \leq |t| \leq 2\mu, \quad |x| + \left| \frac{\xi}{|\xi|} - \xi_0 \right| < \delta_\mu \right\} \cap \Gamma = \emptyset. \end{cases}$$

because  $\rho_0 = (0, (0, \xi_0)) \in \Gamma$ .

In order to state a sufficient condition for (1.1), we define a microlocalized operator of  $P$  at  $\rho_0$  as follows: Let  $h(x)$  be a  $C_0^\infty(\mathbf{R}^n)$  function such that  $0 \leq h \leq 1$ ,  $h(x) = 1$  for  $|x| \leq 1/5$  and  $h(x) = 0$  for  $|x| \geq 7/24$ . For a  $\delta > 0$  we set  $h_\delta(x) = h(x/\delta)$  and  $H_\delta(x, \xi; \lambda) = h_\delta(x)h_\delta(\lambda\xi - \xi_0)$ , where  $0 < \lambda \leq 1$  is a parameter. Let  $\delta_1$  be a small positive such that the projection of  $V$  into  $\mathbf{R}_t \times \mathbf{R}_x^n \times \mathbf{R}_\xi^n$  contains  $\{|t| \leq 2\delta_1\} \times \text{supp } h_{2\delta_1}(x)h_{2\delta_1}(\lambda\xi - \xi_0)$ . For a parameter  $0 < \lambda \leq 1$ , we set

$$P_\lambda = D_t + i h_{\delta_1}(x) Q(t, x, D_x) h_{\delta_1}(\lambda D_x - \xi_0) \equiv D_t + i Q_\lambda(t, x, D_x).$$

THEOREM 1. Let  $\Gamma$  be the above set in Char  $P$  and assume (1.2). Let  $\rho_0 = (0, (0, \xi_0)) \in \Gamma$  and let  $P$  be a pseudodifferential operator of the form (1.3)' in a conic neighborhood  $V$  of  $\rho_0$ . Let  $\delta$  be a small positive such that  $100\delta < \delta_1$  for the above  $\delta_1$ . Assume that for each  $\delta$  there exist non-negative symbols  $\varphi(x, \xi; \lambda) \in S_{1,0}^0$  and  $\alpha(t, x, \xi) \in C^\infty(\mathbf{R}_t; S_{1,0}^0)$  such that  $\{\varphi(x, \xi; \lambda); 0 < \lambda \leq 1\}$  is a bounded set of  $S_{1,0}^0$  and we have

$$(1.8) \quad \begin{cases} \varphi \geq 1 & \text{outside of } \text{supp } H_{5\delta}(x, \xi; \lambda) \\ \varphi = 0 & \text{on } \text{supp } H_\delta(x, \xi; \lambda), \end{cases}$$

$$(1.9) \quad |(H_q \varphi)(t, x, \xi; \lambda)| \leq \alpha(t, x, \xi) \quad \text{on } \{|t| \leq \delta_1\} \times \text{supp } H_{100\delta}(x, \xi; \lambda)$$

and the following estimate: For any  $\varepsilon > 0$  there exists a  $C_\varepsilon > 0$  independent of  $0 < \lambda \leq 1$  such that

$$(1.10) \quad \begin{aligned} & \|u\|^2 + (\log \lambda)^2 \|\alpha(t, x, D_x)u\|^2 \\ & \leq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon (\lambda \|u\|^2 + \lambda^{-2} \|(1 - H_{20\delta}(x, D_x; \lambda)u)\|^2) \end{aligned}$$

if  $u \in C_0^\infty([-\delta_1, \delta_1]; \mathcal{S}(\mathbf{R}_x^n))$ . Then we have (1.1).

COROLLARY. The same conclusion of Theorem 1 follows if we replace (1.9) and (1.10), respectively, by

$$(1.9)' \quad |(H_q \varphi)(t, x, \xi; \lambda)|^2 \leq \alpha(t, x, \xi) \quad \text{on } \{|t| \leq \delta_1\} \times \text{supp } H_{100\delta}(x, \xi; \lambda)$$

and

$$(1.10)' \quad \begin{aligned} & \|u\|^2 + (\log \lambda)^2 \text{Re}(\alpha(t, x, D_x)u, u) \\ & \leq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon (\lambda \|u\|^2 + \lambda^{-2} \|(1 - H_{20\delta}(x, D_x; \lambda)u)\|^2). \end{aligned}$$

REMARK 1. The function  $h(x)$  in Theorem 1 and Corollary is not necessary to be homogeneous spatially. For example, we can replace it by  $h(x_1/\nu)h(x')$  for any  $\nu > 0$  and for  $h(x_1)$ ,  $h(x')$  similar as  $h(x)$ .

REMARK 2. As criteria of hypoellipticity, logarithmic regularity up estimates were used in Morimoto [11-13]. The simple proof of Theorem 1 in the present paper is inspired by Kajitani-Wakabayashi [7; Theorem 1.2] (see also [16]) and Hörmander [6; Lemma 26.9.3].

As an application of Theorem 1, we consider a pseudodifferential operator of principal type which has the following form:

$$(1.11) \quad P = D_t + i a(t, x, D_x)(D_{x_1} + f(t)|D_x|),$$

in a conic neighborhood  $V$  of  $\rho_0 = (0, (0, \xi_0))$ , where  $a(t, x, \xi) \in C^\infty(\mathbf{R}_t; S_{1,0}^0)$ ,  $f(t) \in C^\infty$  satisfy

$$(1.12) \quad a(t, x, \xi) \geq 0, \quad f'(t) > 0 \quad (t \neq 0), \quad f'(0) = 0.$$

It follows from (1.12) that  $P$  satisfies  $(\bar{\Psi})$ .

**THEOREM 2.** *Let  $\rho_0$  be  $(0, (0, \xi_0)) \in T^*(\mathbf{R}^{n+1})$ . Let  $P$  be of the form (1.11) in a conic neighborhood of  $V$  of  $\rho_0$  and satisfy (1.12). Then we have (1.1) if the following conditions are fulfilled;*

$$(1.13) \quad \exists \alpha(t), \beta(t) \in C^\infty; \beta(t) > 0 \quad (t \neq 0), \quad t\alpha'(t) \geq 0,$$

$$(1.14) \quad \beta(t) \leq a(t, x, \xi) \leq \alpha(t) \quad \text{in } V,$$

$$(1.15) \quad |\nabla_x a(t, x, \xi)| + |\nabla_\xi a(t, x, \xi)| |\xi| \leq a(t) \quad \text{in } V,$$

$$(1.16) \quad \lim_{t \rightarrow 0} t\alpha(t) \log f'(t) = 0 \quad \text{and}$$

$$(1.17) \quad \lim_{t \rightarrow 0} t\alpha(t) \log \beta(t) = 0.$$

It follows from (1.13) and (1.14) that  $a(t, x, \xi) > 0$  ( $t \neq 0$ ) and hence we see, together with (1.12),

$$\Gamma \subset \{\tau = t = 0\}$$

Consequently we have (1.2) (and hence (1.7)). The operator of the form (1.11) is infinitely degenerate model corresponding to the case of  $m=0$  in the operator  $P_0$  of Egorov type stated in the introduction. We do not know the microhypoellipticity for a simple operator with  $f(t)$  in (1.11) replaced by  $f(t)x_1^2$ , because of the difficulty in deriving  $L^2$  *a priori* estimate.<sup>†</sup>

However, if  $a(t, x, \xi)$  in (1.12) does not vanish we can treat infinitely degenerate model of  $P_0$  a little more generally. This case is geometrically stated as

$$(1.18) \quad H_1, H_2 \text{ and the radial direction are linearly independent in } V \cap \text{Char } P,$$

which is invariant condition under the multiplication of elliptic factors. If (1.18) is valid then it follows from condition  $(\bar{\Psi})$  that we have the maximal hypoelliptic estimate, in a sense of Helffer-Nourrigat [3], as follows;

$$\|D_t u\|^2 + \|Q_\lambda(t, x, D_x)u\|^2 \leq C(\|P_\lambda u\|^2 + \|u\|^2) \quad (\text{cf., (4.9)}).$$

By means of this estimate, the problem of hypoellipticity for  $D_t + iQ$  can be reduced to the similar one for second operator  $D_t^2 + Q^2$  as in [13]. From now

<sup>†</sup> Some special cases will be studied in the forthcoming paper [17].

on we shall consider the case corresponding to [13; Theorem 4]. Let  $\Gamma$  be a  $C^\infty$  submanifold of codimension 2 in  $\text{Char } P$  and symplectic, that is,

$$(1.19) \quad T\Gamma \cap T\Gamma^\perp = 0 \text{ at every point of } \Gamma.$$

It follows from (1.19) that both  $H_1$  and  $H_2$  are transversal to  $\Gamma$  because  $H_1, H_2 \in T(\text{Char } P)^\perp \subset T\Gamma^\perp$ . Hence (1.2) holds and so (1.7). In order to state the additional condition we have to fix a special coordinate. By a symplectic linear transformation, it follows from (1.18) that  $q(t, x, \xi)$  of (1.3) satisfies  $\partial_{\xi_1} q(t, x, \xi) \neq 0$ . It follows from the implicit function theorem that there exist  $a(t, x, \xi) \in C^\infty(\mathbf{R}_t; S_{1,0}^0)$  and  $b(t, x, \xi') \in C(\mathbf{R}_t; S_{1,0}^1)$  such that

$$(1.20) \quad q(t, x, \xi) = a(t, x, \xi)(\xi_1 + b(t, x, \xi')), \quad a(t, x, \xi) \neq 0, \quad \text{in } V,$$

where  $\xi' = (\xi_2, \dots, \xi_n)$ . Let  $\gamma_\rho(t)$  be the bicharacteristic of  $p_1 = \tau$  through  $\rho = (0, x, 0, \xi_1, \xi') \in V \cap \text{Char } P$ , that is,  $\gamma_\rho(t) = \{(t, x, 0, \xi_1, \xi')\}$ . By setting  $\xi_1 = -b(t, x, \xi')$  we define a projection  $\pi\gamma_\rho(t)$  into  $\text{Char } P$  of the bicharacteristic. We assume that

$$(1.21) \quad \begin{cases} \text{there exist a } \delta_0 > 0 \text{ and a } 0 \neq e(t, x, \xi) \in C^\infty(\mathbf{R}_t; S_{1,0}^0) \\ \text{such that for any } \rho = (0, x, 0, \xi) \in V \cap \text{Char } P, F_\rho(t) \equiv \\ (e\partial_t q)|_{\text{Char } P}(\pi\gamma_\rho(t)) \text{ has a unique extremum at } t = s(\rho) \\ \text{in } (-\delta_0, \delta_0), \text{ and } s(\rho) \text{ belongs to } C^\infty \text{ with respect to } \rho. \end{cases}$$

**THEOREM 3.** *Let  $\rho_0$  be  $(0, (0, \xi_0)) \in \Gamma$  and let  $P$  be the form (1.3)' and satisfy (1.18) in a conic neighborhood  $V$  of  $\rho_0$ . Assume that  $\Gamma$  is a  $C^\infty$ -symplectic submanifold and of codimension 2 in  $\text{Char } P$ . Then we have (1.1) if the condition (1.21) holds with  $q(t, x, \xi)$  expressed as (1.20).*

As a typical example of Theorem 3 we have the following:

$$p(t, x, \tau, \xi) = \tau + i \left\{ \xi_1 + \int_0^t \exp(-(s^2 + x_1^2)^{-\delta/2}) ds |\xi| \right\}, \quad \delta > 0.$$

## 2. Proof of Theorem 1

Let  $\chi(t)$  be a  $C_0^\infty(\mathbf{R}_t)$  function such that  $0 \leq \chi(t) \leq 1$ ,  $\chi(t) = 1$  for  $|t| \leq 1$ ,  $\chi(t) = 0$  for  $|t| \geq 2$ . Set  $\Phi(\tau, \xi; \mu) = \chi(|\tau|/\mu|\xi|)(1 - \chi(|\xi|))$  for a small  $\mu > 0$ . For cutting  $\mathbf{R}_\xi^n$  we define the following:

**DEFINITION 1.** For  $\delta > 0$  and  $\xi_0 \in \mathbf{R}^n$  ( $|\xi_0| = 1$ ) we say that a function  $\phi(\xi) \in C^\infty(\mathbf{R}^n)$  belongs to  $\Psi_{\delta, \xi_0}$  if  $0 \leq \phi \leq 1$  satisfies

$$\begin{cases} \phi(\xi)=1 & \text{for } |\xi/|\xi|-\xi_0|\leq\delta/12 \text{ and } |\xi|\geq 2/3, \\ \phi(\xi)=0 & \text{for } |\xi/|\xi|-\xi_0|\geq\delta/10 \text{ or } |\xi|\leq 1/2, \\ \phi(\xi)=\phi(\xi/\lambda) & \text{for } 0<\lambda\leq 1 \text{ and } |\xi|\geq 1. \end{cases}$$

In the proof of the theorem we may assume  $u\in\mathcal{E}'$  and hence  $u$  belongs to  $H_{-N}$  for an integer  $N>0$ . Suppose that  $\rho_0\notin\text{WF}(Pu)$ . Then for a sufficiently small  $\mu>0$  we have

$$\chi(t/2\mu)\Phi(D_t, D_x; 2\mu)\phi_\mu(D_x)h_\mu(x)Pu\in H_s$$

for any real  $s$ , where  $\phi_\mu(\xi)\in\Psi_{\mu,\xi_0}$ . If we set  $v=\chi(t/\mu)\Phi(D_t, D_x; \mu)u$  then it follows from (1.7) that  $\phi_{\delta_\mu}(D_x)h_{\delta_\mu}(x)Pv\in H_s$  for a  $\phi_{\delta_\mu}(\xi)\in\Psi_{\delta_\mu,\xi_0}$  because  $P$  is microhypoelliptic on the intersection of  $\text{supp } h_{\delta_\mu}(x)\phi_{\delta_\mu}(\xi)$  and the support of derivatives of  $\chi(t/\mu)\Phi(\tau, \xi; \mu)$ . Fix a positive  $\delta$  such that  $100\delta<\min(\delta_\mu, \delta_1)$ . We shall show  $\phi_\delta(D_x)h_\delta(x)v\in H_s$ , which will yield (1.1).

For the above  $\delta$  we take  $\varphi(x, \xi; \lambda)$  in the assumption of the theorem. For an integer  $l>s+N+1$  we denote a pseudodifferential operator with a symbol  $\lambda^{l\varphi(x, \xi; \lambda)}$  by  $K(x, D_x; \lambda)$ . If  $\lambda$  varies  $0<\lambda\leq 1$  then  $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)$  belongs to a bounded set of  $S_{1,\varepsilon_1}^0$  for any small  $\varepsilon_1>0$ . For any real  $a$ ,  $[K(x, D_x; \lambda), H_{10\delta}(x, D_x; \lambda)]\lambda^{-a}$  belongs to a bounded set of  $S_{1,0}^{(l-a-\varepsilon_1)}$  because of (1.8). Furthermore,  $h_{10\delta}(x)$ ,  $h_{10\delta}(\lambda D_x - \xi_0)$  and  $K(x, D_x; \lambda)$  are commutative, each other, as a product of three factors, neglecting term in  $\lambda^a \times S_{1,0}^{(l-a-\varepsilon_1)}$ .

Let  $w\in\mathcal{S}$  satisfy

$$(2.1) \quad \text{supp } w \subset \{|t|\leq 2\mu\}$$

and substitute  $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)w$  into (1.10) in place of  $u$ . Then

$$\begin{aligned} & \|KH_{10\delta}w\|^2 + (\log \lambda)^2 \|\alpha(t, x, D_x)KH_{10\delta}w\|^2 \\ (2.2) \quad & \leq 2\varepsilon \{ \|h_{10\delta}(\lambda D_x - \xi_0)h_{10\delta}(x)KPw\|^2 + \|H_{10\delta}[Q(t, x, D_x), K]w\|^2 \} \\ & + C_\varepsilon(\lambda \|KH_{10\delta}w\|^2 + \lambda^{2s+1} \|w\|_{(0, -N)}^2) \end{aligned}$$

because the same commutative argument as above follows for  $H_{10\delta}$  and  $KQ$  by means of (1.8). Here for real  $a$  we have set  $\|w\|_{(0, a)} = \|(1+A)^a w\|$ ,  $A^2=1+|D_x|^2$  and by this norm we define the space  $H_{(0, a)}$ . Note that the principal symbol of  $[Q, K]$  is equal to

$$-il(\log \lambda)(H_q\varphi)\lambda^{l\varphi(x, \xi; \lambda)}$$

and symbols of lower orders are a sum of  $\lambda^{1/2+l\varphi(x, \xi; \lambda)}$  multiplied by symbols in a bounded set of  $S_{1,0}^0$  uniformly with respect to  $0<\lambda\leq 1$ . It follows from (1.9) that



$$(2.3) \quad \|H_{10\delta}[Q(t, x, D_x), K]w\|^2 \leq l^2(\log \lambda)^2 \|\alpha(t, x, D_x)KH_{10\delta}w\|^2 \\ + C_l(\lambda \|KH_{10\delta}w\|^2 + \lambda^{2s+1} \|w\|_{(0, -N)}^2).$$

Choose  $2\varepsilon l^2 < 1$ , then for a constant  $C'_l$  we have

$$(1 - 2\varepsilon \lambda C_l) \|KH_{10\delta}w\|^2 \\ \leq 2\varepsilon \|h_{10\delta}(\lambda D_x - \xi_0)h_{10\delta}(x)KPw\|^2 + C'_l \lambda^{2s+1} \|w\|_{(0, -N)}^2.$$

It follows from (1.8) that  $\|h_\delta(\lambda D_x - \xi_0)h_\delta(x)w\|^2 \leq \|KH_{10\delta}w\|^2 + \tilde{C}_l \lambda^{2s+1} \|w\|_{(0, -N)}^2$ . Take a  $\lambda_0$  satisfying  $\lambda_0(2\varepsilon C_l + C_\varepsilon) < 1/4$ . Then for  $0 < \lambda \leq \lambda_0$  we have

$$\|h_\delta(\lambda D_x - \xi_0)h_\delta(x)w\|^2 \\ \leq 4\varepsilon \{ \|h_{10\delta}(\lambda D_x - \xi_0)h_{10\delta}(x)Pw\|^2 + C''_l \lambda^{2s+1} \|w\|_{(0, -N)}^2 \}.$$

Multiplying  $\lambda^{-2s}(1 + \kappa \lambda^{-1})^{-2(l+1)}$  with a parameter  $\kappa > 0$  by both sides, for  $0 < \lambda \leq \lambda_0$  we have

$$\|h_\delta(\lambda D_x - \xi_0)(1 + \kappa \lambda)^{-(l+1)}h_\delta(x)w\|_s^2 \\ \leq 4\varepsilon (\|h_{10\delta}(\lambda D_x - \xi_0)(1 + \kappa \lambda)^{-(l+1)}h_{10\delta}(x)Pw\|_s^2 + C'_l \lambda \|w\|_{(0, -N)}^2)$$

because  $\lambda^{-1}$  is equivalent to  $|\xi|$  on  $\text{supp } h(\lambda \xi - \xi_0)$ . Integrate  $\lambda$  from 0 to  $\lambda_0$  after dividing both sides by  $\lambda$ . Then by means of [12; Proposition 1.7] we have for suitable  $\phi_\delta(\xi) \in \mathcal{P}_{\delta, \xi_0}$  and  $\tilde{\phi}_\delta(\xi) \in \mathcal{P}_{\gamma_0 \delta, \xi_0}$ ,

$$\|(1 + \kappa \lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)w\|_{(0, s)}^2 \\ \leq C(\|(1 + \kappa \lambda)^{-(l+1)}\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pw\|_{(0, s)}^2 + \|w\|_{(0, -N)}^2).$$

It follows from  $u \in H_{-N}$  that one can find a sequence  $\{u_j\}$  in  $\mathcal{S}$  satisfying  $u_j \rightarrow u \in H_{-N}$ . If  $w_j = \chi(t/\mu)\Phi(D_t, D_x; \mu)u_j$  then  $w_j \rightarrow v$  in  $H_{(0, -N)}$  and  $Pw_j \rightarrow Pv$  in  $H_{(0, -(N+1))}$ . Letting  $j \rightarrow \infty$  in the above estimate with  $w = w_j$ , in view of  $Pv \in H_s$  we get for  $\kappa > 0$

$$\|(1 + \kappa \lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)v\|_{(0, s)}^2 \leq C(\|\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pv\|_s^2 + \|u\|_{-N}^2).$$

Making  $\kappa \rightarrow 0$  we see  $\phi_\delta(D_x)h_\delta(x)v \in H_s$  because  $v = \chi(t/\mu)\Phi(D_t, D_x; \mu)u$ . Thus we have proved that  $Pu \in H_s$  at  $\rho_0$  implies  $u \in H_s$  at  $\rho_0$ .

The proof of Corollary is obvious if we replace the term  $\|\alpha(t, x, D_x)KH_{10\delta}w\|^2$  in (2.2) and (2.3) by  $\text{Re}(\alpha(t, x, D_x)KH_{10\delta}w, KH_{10\delta}w)$ .

### 3. Proof of Theorem 2

If  $\xi_0 \notin \Sigma = \{\xi_1 + f(0) | \xi| = 0\}$  then the theorem is obvious because  $q(t, x, \xi) = a(\xi_1 + f(t)|\xi|)$  is semi-definite in a small conic neighborhood of  $\rho_0 = (0, (0, \xi_0))$  and we can apply the result about the propagation of regularities (Hörmander

[6; Proposition 26.6.1]). In what follows we assume  $\xi_0 \in \Sigma$  (though we will not use this condition). We apply Theorem 1 by setting

$$\varphi(x, \xi) = (1 - h_{\delta\delta}(x)) + (1 - h_{\delta\delta}(\lambda\xi - \xi_0)).$$

Then we have (1.8) and it follows from (1.14) and (1.15) that (1.9) holds with  $\alpha(t, x, \xi) = C\alpha(t)$  for a suitable  $C > 0$  if  $\delta$  is small enough. The proof of Theorem 2 would be completed if we could show (1.10). Set

$$a_\lambda(t, x, \xi) = h_{\delta_1}(x) a(t, x, D_x) h_{2\delta_1}(\lambda D_x - \xi_0).$$

Let  $A(t) = a_\lambda^\psi(t, x, D_x)$  denote a pseudodifferential operator with Weyl symbol  $a_\lambda(t, x, \xi)$ . Setting  $B(t) = (D_{x_1} + f(t)|D_x|) h_{\delta_1}(\lambda D_x - \xi_0)$  moreover, we consider  $A(t), B(t)$  as a real operator on Hilbert space  $\mathcal{H} = L^2(\mathbf{R}_x^n)$ . Note that for a fixed  $\lambda > 0$   $B(t)$  is bounded operator in  $\mathcal{H}$ . If  $\mathcal{Q}_+(t) = \{\xi; \xi_1 + f(t)|\xi| > 0\}$  and if

$$S_+(t)v(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} 1_{\mathcal{Q}_+(t)}(\xi) \hat{v}(\xi) d\xi, \quad v \in \mathcal{H}, \quad S_-(t) = Id - S_+(t)$$

then we can define the sign  $M(t)$  of  $B(t)$ ,  $M(t) = S_+(t) - S_-(t)$  and it follows from (1.12) that

$$(3.1) \quad (M(t_1) - M(t_2))(t_1 - t_2) \geq 0 \quad \text{on } \mathcal{H}.$$

From this condition we have the following lemma given by Lerner [8; §2]:

LEMMA (Lerner [8, 9]). *There exists a  $\delta' > 0$  independent of  $0 < \lambda \leq 1$  such that for any  $u(t) \in C_0^\infty(\mathbf{R}_t; \mathcal{H})$  we have*

$$(3.2) \quad 2 \int |P_\lambda u(t)|_{\mathcal{H}} dt \geq \sup |u(t)|_{\mathcal{H}} \quad \text{if } \text{supp } u \subset \{|t| \leq \delta'\},$$

where  $|\cdot|_{\mathcal{H}} = \|\cdot\|_{L^2(\mathbf{R}_x^n)}$ .

PROOF. By means of [8; Lemma 2.3.1], it follows from (3.1) that

$$(3.3) \quad \text{Re} \int (\dot{u}(t), M(t)u(t))_{\mathcal{H}} dt \leq 0, \quad \dot{u}(t) = \frac{du}{dt}(t).$$

If  $H(t)$  denotes Heaviside function then for any  $T$  we have

$$\begin{aligned} & -\text{Re} \int (\dot{u}(t), \{H(t-T)S_+(t) - H(T-t)S_-(t)\} u(t))_{\mathcal{H}} dt \\ (3.4) \quad & = -\text{Re} \int (\dot{u}(t), H(t-T)(M+S_-)u(t) + H(T-t)(M-S_+)u(t))_{\mathcal{H}} dt \\ & \geq -\text{Re} \int (\dot{u}(t), \{H(t-T)S_-(t) - H(T-t)S_+(t)\} u(t))_{\mathcal{H}} dt, \end{aligned}$$

where we have used (3.3) in the last inequality. Adding the left hand side of

(3.4) to both sides of (3.4), we have in view of  $S_+ + S_- = Id$

$$\begin{aligned}
 & -2\operatorname{Re} \int (\dot{u}(t), \{H(t-T)S_+(t) - H(T-t)S_-(t)\}u(t))_{\mathcal{H}} dt \\
 (3.5) \quad & \geq -\operatorname{Re} \int (\dot{u}(t), \{H(t-T) - H(T-t)\}u(t))_{\mathcal{H}} dt \\
 & = 2\|u(T)\|_{\mathcal{H}}^2.
 \end{aligned}$$

It follows from [8; Lemma 2.3.2] that

$$(3.6) \quad \operatorname{Re}(\pm S_{\pm} \operatorname{Re}(AB)) \geq -\frac{10}{3} \|A\|^{1/4} \| [A, B] \|^{1/2} \| [B, [B, A]] \|^{1/4},$$

where  $\|A\|$  denotes the operator norm of  $A(t)$  in  $\mathcal{H}$ . Note that the right hand side of (3.6) has the bound independent of  $\lambda$ . Since the difference between  $P_{\lambda}$  and  $D_t + i \operatorname{Re}(A(t)B(t))$  is bounded in  $\mathcal{H}$  uniformly with respect to  $0 < \lambda \leq 1$ , in view of (3.5) and (3.6) there exists a  $C > 0$  independent of  $\lambda$  such that

$$\begin{aligned}
 & \operatorname{Re} \int (P_{\lambda} u(t), i \{H(t-T)S_+(t) - H(T-t)S_-(t)\}u(t))_{\mathcal{H}} dt \\
 & \geq \|u(T)\|_{\mathcal{H}}^2 - C \int |u(t)|_{\mathcal{H}}^2 dt.
 \end{aligned}$$

If  $\operatorname{supp} u \subset \{|t| \leq \delta'\}$  then the second term of the right hand side is estimated above from  $2C\delta' \sup |u(t)|_{\mathcal{H}}^2$ , so that we have (3.2) for a small  $\delta' > 0$  satisfying  $4C\delta' \leq 1$ .

By means of the Schwartz inequality it follows from (3.2) that

$$(3.7) \quad \|P_{\lambda} u\| \geq (2\delta')^{-1} \|u\| \quad \text{if } \operatorname{supp} u \subset \{|t| \leq \delta'\}.$$

It follows from (1.16) and (1.17) that for any  $\varepsilon > 0$  there exists a  $\delta_{\varepsilon} > 0$  such that

$$(3.8) \quad t\alpha(t)\{|\log f'(t)| + |\log \beta(t)|\} \leq \varepsilon^2 \quad \text{if } |t| \leq \delta_{\varepsilon}.$$

For the sake of simplicity we assume  $\alpha(t)$  is even function (the general case would be clear once we could prove this case). It follows from the monotonicity of  $\alpha(t)$  that for a small parameter  $\lambda > 0$  there exists a unique  $t_{\lambda} > 0$  such that  $t_{\lambda}\alpha(t_{\lambda})|\log \lambda| = 2\varepsilon$ . Similarly we choose  $s_{\lambda} > 0$  such that  $s_{\lambda}\alpha(s_{\lambda})|\log \lambda| = \varepsilon$ . For a while we assume  $\lambda$  is sufficiently small such that  $s_{\lambda} < \delta_{\varepsilon}$ . If we set  $\delta' = t_{\lambda}$  in (3.7) then

$$\begin{aligned}
 (3.9) \quad & 4\varepsilon \|P_{\lambda} u\| \geq \|\alpha(t_{\lambda})(\log \lambda)u\| \\
 & \geq \|\alpha(t)(\log \lambda)u\| \quad \text{if } \operatorname{supp} u \subset \{|t| \leq t_{\lambda}\}.
 \end{aligned}$$

If  $s_{\lambda} \leq |t| \leq \delta_{\varepsilon}$  then it follows from (3.8) that

$$\frac{\varepsilon}{|\log \lambda|} \{ |\log f'(t)| + |\log \beta(t)| \} \leq \varepsilon^2,$$

so that if  $0 < \lambda \leq \lambda_\varepsilon$  for a sufficiently small  $\lambda_\varepsilon$  then

$$(3.10) \quad f'(t), \beta(t) \geq \lambda^\varepsilon \quad \text{on } s_\lambda < |t| \leq \delta_1.$$

In fact, if  $\lambda_\varepsilon$  is small enough we have  $f'(t), \beta(t) \geq (\lambda_\varepsilon)^\varepsilon$  for  $\delta_\varepsilon < |t| \leq \delta_1$  in view of (1.12) and (1.13). Note that

$$(3.11) \quad \begin{aligned} \|P_\lambda u\|^2 &= \|D_t u\|^2 + \|a_\lambda(t, x, D_x)Bu\|^2 \\ &+ 2 \operatorname{Re}((\partial_t a_\lambda)Bu, u) \\ &+ 2 \operatorname{Re}(a_\lambda f'(t) |D_x| h_{\delta_1}(\lambda D_x - \xi_0)u, u) \end{aligned}$$

Since it follows from (3.10) and (1.14) that

$$a_\lambda(t, x, \xi) \geq \lambda^\varepsilon \quad \text{on } \{s_\lambda \leq |t| \leq \delta_1\} \times \operatorname{supp} H_{\delta_1}(x, \xi; \lambda)$$

the second term of the right hand side of (3.11) is estimated above from

$$C(\lambda^{-\varepsilon} \|a_\lambda Bu\| \|u\| + \|u\|^2) \leq \|a_\lambda Bu\|^2 + C' \lambda^{-2\varepsilon} \|u\|^2.$$

By means of (3.10) again we have, if  $\operatorname{supp} u \subset \{s_\lambda \leq |t| \leq \delta_1\}$ ,

$$2 \operatorname{Re}(a_\lambda f'(t) |D_x| h_{\delta_1}(\lambda D_x - \xi_0)u, u) \geq \lambda^{2\varepsilon-1} \|H_{20\delta} u\|^2 - C \|u\|^2.$$

Therefore, if  $\operatorname{supp} u \subset \{s_\lambda \leq |t| \leq \delta_1\}$  then

$$\|P_\lambda u\|^2 \geq \lambda^{2\varepsilon-1} \|H_{20\delta} u\|^2 - C \lambda^{-2\varepsilon} \|u\|^2,$$

provided that  $0 < \lambda \leq \lambda_\varepsilon$ . If  $\varepsilon < 1/16$  and if  $0 < \lambda \leq \min(\lambda_\varepsilon, \varepsilon^2) = \lambda'_\varepsilon$  we have

$$(3.12) \quad \begin{aligned} \varepsilon \|P_\lambda u\|^2 &\geq \lambda^{2\varepsilon-1/2} \|u\|^2 - C \lambda^{-2} \|(1 - H_{20\delta})u\|^2 \\ &\quad \text{if } \operatorname{supp} u \subset \{s_\lambda \leq |t| \leq \delta_1\}. \end{aligned}$$

Let  $\chi_0(t)$  be  $C^\infty(\mathbf{R})$  such that  $\chi_0(t) = 1$  for  $t \leq 0$  and  $\chi_0(t) = 0$  for  $t \geq 1$ . Set  $\phi_\pm(t) = \chi_0(\pm(t \pm s_\lambda)/(s_\lambda - t_\lambda))$  and  $\phi(t) = \phi_+(t)\phi_-(t)$ . The fact that  $t_\lambda - s_\lambda \geq c\varepsilon/|\log \lambda|$  for a suitable  $c > 0$  shows  $|\phi^{(j)}(t)| \leq C_\varepsilon |\log \lambda|^j$  ( $j = 1, 2, \dots$ ). It follows from (3.12) that

$$(3.13) \quad \begin{aligned} \|[P_\lambda, \phi]u\|^2 &= \|\phi' u\|^2 \\ &\leq C \lambda^{1/2-2\varepsilon} \{ \|[P_\lambda, \phi']u\|^2 + |\log \lambda|^2 (\|P_\lambda u\|^2 + \lambda^{-2} \|(1 - H_{20\delta})u\|^2) \}. \end{aligned}$$

Since similar estimates hold with  $\phi$  replaced by  $\phi^{(j)} |\log \lambda|^{-j}$ ,  $j = 1, 2, \dots$ , in view of  $u = \phi(t)u + (1 - \phi(t))u$ , it follows from (3.9) and (3.12) that

$$(3.14) \quad 16\varepsilon \|P_\lambda u\|^2 \geq \|\alpha(t)(\log \lambda)u\|^2 - C \lambda^{-2} \|(1 - H_{20\delta})u\|^2.$$

if  $0 < \lambda \leq \lambda'_\varepsilon$ . From (3.14), (3.7) and (3.12) we have the desired estimate (1.10)

because it is trivial for  $\lambda'_\varepsilon < \lambda \leq 1$  by taking a sufficiently large  $C_\varepsilon$  in the right hand side.

#### 4. Proof of Theorem 3

Since  $\rho_0 = (0, (0, \xi_0)) \in \Gamma$  it follows from (1.20) that  $\xi_{01} + b(0, 0, \xi'_0) = 0$ . By taking the canonical transformation such that  $\xi_1 + b(0, 0, \xi') \rightarrow \xi_1$  and  $\xi' \rightarrow \xi'$  we may assume that  $\xi_0 = (0, \xi'_0)$ ,  $|\xi'_0| = 1$ . Because  $\Gamma$  is of codimension 2 in  $\text{Char } P$  it follows from (1.20) and (1.6) that  $\partial_t b(t, x, \xi')$  has the definite sign. Note that

$$(4.1) \quad (a^{-1}\partial_t q)|_{\text{Char } P}(\pi\gamma_\rho(t)) = \partial_t b(t, x, \xi').$$

For each  $\rho = (0, x, (0, -b(t, x, \xi'), \xi')) \in \text{Char } P \cap V$ , let  $t(x, \xi')$  denote the extremal point in the condition (1.21). Since it follows from (4.1) that  $F_\rho(t)$  in (1.21) equals  $(\partial_t b)(t, x, \xi')$  for some  $\tilde{t}(t, x, \xi') \in C^\infty(\mathbf{R}_t \times \mathbf{R}_{x_1}; S_{1,0}^0)$ , we have in a conic neighborhood of  $\rho_0$

$$(4.2) \quad \begin{aligned} |(\partial_t b)(t(x, \xi'), x, \xi')| &< |(\partial_t b)(s, x, \xi')| \leq |(\partial_t b)(t, x, \xi')| \\ &\text{if } 0 < |s - t(x, \xi')| < |t - t(x, \xi')|. \end{aligned}$$

Set  $\tilde{b}(t, x, \xi') = \int_{t(x, \xi')}^t \partial_t b(s, x, \xi') ds$  and take the canonical transformation in  $T^*(\mathbf{R}_x^n)$ , keeping  $x_1$  variable, such that

$$\xi_1 + b(t(x, \xi'), x, \xi') \longrightarrow \xi_1 \quad (\text{and } (0, \xi'_0) \rightarrow (0, \xi'_0)).$$

Then  $\xi_1 + b(t, x, \xi')$  is transformed to  $\xi_1 + b_0(t, x, \xi')$  of the form:

$$(4.3) \quad b_0(t, x, \xi') = \tilde{b}(t, x_1, \Phi(x, \xi'), \Psi(x, \xi')) \text{ in a small conic neighborhood of } \rho_0,$$

where  $\Phi(x, \xi') \in S_{10}^0$ ,  $\Psi(x, \xi') \in S_{1,0}^1$ . It follows from (4.2) that

$$(4.4) \quad |\nabla_x b_0(t, x, \xi')| + |\nabla_{\xi'} b_0(t, x, \xi')| |\xi| \leq C |\partial_t b_0(t, x, \xi')|.$$

In fact, for example, the direct calculation gives

$$(4.5) \quad \begin{aligned} |\partial_{x_2} b_0(t, x, \xi')| &\leq C_1 |\partial_t b(t(x_1, x', \xi'), x', \xi')|_{(x', \xi') = (\Phi(x, \xi'), \Psi(x, \xi'))} \\ &+ C_2 \left| \int_{t(x, \xi')}^t |\partial_t \partial_{x_2} b(s, x_1, x', \xi')|_{(x', \xi') = (\Phi(x, \xi'), \Psi(x, \xi'))} ds \right|. \end{aligned}$$

By means of (4.2), the first term of the right hand side is estimated above from  $C |\partial_t b_0(t, x, \xi')|$ . Because  $\partial_t b$  is semi-definite we have  $|\partial_t \partial_{x_2} b| \leq C |\partial_t b|^{1/2}$  and the second term is estimated above from

$$C \left| \int_{t(x, \xi')}^t |(\partial_t b)(s, x, \xi')|^{1/2} ds \right| \leq C' |\partial_t b(t, x, \xi')|^{1/2}$$

with  $(x', \xi') = (\Phi(x, \xi'), \Psi(x, \xi'))$ . Here we have used (4.2) in the last inequality.

As stated in the section 1, it follows from (1.19) that Hamilton vector fields  $H_1 = \partial_t$  and  $H_2 = H_q$  are transversal to  $\Gamma$ . In view of (4.4), the fact that  $\partial_t b_0(0, 0, \xi'_0) = 0$  shows that

$$(4.6) \quad \left\{ \begin{array}{l} \text{for any small } \mu > 0 \text{ there exists a } \delta_\mu > 0 \text{ such that} \\ \left\{ (t, x, 0, \xi); \mu \leq \max(|t|, |x_1|) \leq 2\mu, |x'| + \left| \frac{\xi}{|\xi|} - \xi_0 \right| < \delta_\mu \right\} \cap \Gamma = \emptyset. \end{array} \right.$$

In the new variable we shall apply Corollary of Theorem 1, together with Remark 1. For the brevity we write  $b$  instead of  $b_0$  in what follows. Set  $\varphi(x, \xi) = (1 - \chi(x_1/\mu)) + (1 - h_{2\delta}(x')) + (1 - h_{5\delta}(\lambda\xi - \xi_0))$ . Choosing  $\nu = 2\mu/\delta$  in Remark 1 of Corollary we have (1.8). Since  $H_q\varphi = a(\partial_{x_1}\varphi + H_b\varphi) + (H_a\varphi)(\xi_1 + b)$ , in view of  $a \neq 0$  it follows from (4.4) and (1.20) that

$$(4.7) \quad \begin{aligned} |H_q\varphi|^2 &\leq C((\chi'(x_1/\mu))^2 + a\partial_t b/|\xi| + (q/|\xi|)^2) \\ &\quad \text{on } \{|t| \leq \delta_1\} \times \text{supp } H_{100\delta}(x, \xi; \lambda) \end{aligned}$$

because the second term of the right hand side is non-negative by means of (4.1) and (1.6). Putting  $\alpha(t, x, \xi)$  equal to the right hand side of (4.7), we shall check (1.10)'. It follows from (4.6) that

$$(4.8) \quad \lambda^{-1} \|\chi'(x_1/\mu) H_{20\delta} u\|^2 \leq C(\|P_\lambda u\|^2 + \|u\|^2).$$

Setting  $Q_\lambda(t, x, \xi) = Q(t, x, \xi) H_{\delta_1}(x, \xi; \lambda)$  we have

$$\|P_\lambda u\|^2 = \|D_t u\|^2 + \|Q_\lambda(t, x, D_x)u\|^2 + 2 \operatorname{Re} (Op(\partial_t Q_\lambda(t, x, \xi))u, u),$$

where  $Op(r)$  denotes the pseudodifferential operator with symbol  $r$ . Since the principal symbol of  $\partial_t Q_\lambda(t, x, \xi)$  equals  $(a\partial_t b + (\partial_t a/a)q)H_{\delta_1}(x, \xi; \lambda)$  it follows from the Schwartz inequality

$$(4.9) \quad \begin{aligned} \|P_\lambda u\|^2 &\geq \|D_t u\|^2 + \|Q_\lambda(t, x, D_x)u\|^2/2 \\ &\quad + 2 \operatorname{Re} (Op(a\partial_t b H_{\delta_1})u, u) - C\|u\|^2 \\ &\geq \|D_t u\|^2 + \|Q_\lambda(t, x, D_x)u\|^2/2 - C'\|u\|^2. \end{aligned}$$

Noting that  $(a\partial_t b/|\xi| + (q/|\xi|)^2)H_{20\delta}^2 \leq a\partial_t b H_{\delta_1}/\lambda + Q_\lambda^2/\lambda^2$ , by means of the sharp Gårding inequality we have (1.10)' from (4.8) and (4.9), because it follows from the Poincaré inequality that the term  $\|u\|^2$  is absorbed by  $\|D_t u\|^2$  if  $\delta_1$  is small enough.

## References

- [1] Beals, R. and Fefferman, C., On local solvability of linear partial differential equations, Ann. of Math. 97 (1973), 482-498.
- [2] Egorov, Yu. V., Subelliptic operators, Russian Math. Surveys 30: 2 (1975), 59-118,

- 30: 3 (1975), 55-105.
- [3] Helffer, B. and Nourrigat, J., Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs, Progress Math. Vol. 58 Boston: Birkhäuser 1985.
  - [4] Hörmander, L., Pseudo-differential operators and non-elliptic boundary problems, Ann. of Math. 83 (1966), 129-209.
  - [5] Hörmander, L., Subelliptic operators, Seminar on Singularities of Solution of Linear Partial Differential Equations, Princeton University Press 1979, 3-49.
  - [6] Hörmander, L., The analysis of Linear Partial Differential Operators IV, (1985), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.
  - [7] Kajitani, K. and Wakabayashi, S., Propagation of singularities for several classes of pseudodifferential operators, Bull. Sc. Math. 2<sup>e</sup> série 115 (1991), 397-449.
  - [8] Lerner, N., Sufficiency of condition  $(\Psi)$  for local solvability in two dimensions, Ann. of Math. 128 (1988), 243-258.
  - [9] Lerner, N., An iff solvability condition for the oblique derivative problem, Séminar EDP 90-91, Ecole Polytechnique, exposé n° 18.
  - [10] Lerner, N., Nonsolvability in  $L^2$  for a first-order operator satisfying condition  $(\Psi)$ , Ann. of Math. 139 (1994), 363-393.
  - [11] Morimoto, Y., Criteria for hypoellipticity of differential operators, Publ. RIMS Kyoto Univ. 22 (1986), 1129-1154.
  - [12] Morimoto, Y., A criterion for hypoellipticity of second order differential operators, Osaka J. Math. 24 (1987), 651-675.
  - [13] Morimoto, Y., Hypoelliptic operators in  $\mathbb{R}^3$  of the form  $X_1^2 + X_2^2$ , J. Math. Kyoto Univ. 32 (1992), 461-484.
  - [14] Moyer, R.D., Local solvability in two dimensions: Necessary conditions for the principal-type case, Mimeographed manuscript, University of Kansas 1978.
  - [15] Nirenberg, L. and Treves, F., On local solvability of linear partial differential equations, I. Necessary conditions, II. Sufficient conditions, Correction. Comm. Pure Appl. Math. 23 (1970), 1-38 and 459-509; 24 (1971), 279-288.
  - [16] Wakabayashi, S. and Suzuki, M., Microhypoellipticity for a class of pseudodifferential operators with double characteristics, Funkciaj Ekvacioj, 36 (1993), 519-556.
  - [17] Morimoto, Y., Local solvability and hypoellipticity for pseudodifferential operators of Egorov type with infinite degeneracy, to appear in Nagoya Math. J., 139 (1995).